

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230 Tutorial 1
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0.1 Complex Numbers and Basic Algebraic Manipulations

We usually denote $\mathbb{C} = \{a + bi : a \text{ and } b \text{ are real numbers}\} = \text{All complex numbers}$.

Definition 1. If $z \in \mathbb{C}$ and $z = a + bi$, then a is called the **real part** of z and is denoted by $\text{Re}(z)$. Similarly, b is called the **imaginary part** of z and is denoted by $\text{Im}(z)$. We say z is **real** if $b = 0$ and z is **purely imaginary** if $a = 0$.

Remark: Be careful that both $\text{Re}(z)$ and $\text{Im}(z)$ are real numbers !!

Proposition 1. (Algebraic Operation) Suppose that z , w and u are complex numbers, then the followings hold:

- Commutative Law (addition): $z + w = w + z$;
- Associative Law (addition): $z + (w + u) = (z + w) + u$;
- Commutative Law (multiplication): $zw = wz$;
- Associative Law (multiplication): $z(wu) = (zw)u$;
- Distributive Law : $z(w + u) = zw + zu$.

Remark: Actually the operation of addition and multiplication is the same as the one of real number. The above properties can be proved by direct computation.

Definition 2. For any complex number $z = a + bi$, we define the conjugate of z to $a - bi$ and is denoted by \bar{z} .

Remark: It is just a reflection along x -axis (real axis).

Theorem 1. (Euler's Formulae) For any complex number $z = a + bi$, we have

$$e^z = e^{a+bi} = e^a e^{bi} = e^a (\cos b + i \sin b)$$

Remark: It can be easily proved by Taylor expansion of e^x .

If we write $z = a + bi$, we would said that this complex number is expressed in **standard form**. From the powerful Eulers Formulae, we are motivated to consider the polar form of complex number.

Proposition 2. If we write $z = a + bi$, the **polar form** of z is given by $z = |z|e^{i\theta}$, where $|z| = \sqrt{a^2 + b^2}$ and $\tan \theta = b/a$. We called $|z|$ the **modulus** or **length** of z . We called θ the **argument** of z .

Remark 1 : If θ_0 is the argument of z , then $\theta_0 + 2\pi$ is also the argument of z due to the periodicity of tan function. To avoid writing multi-value of argument of z , if we restrict argument θ in $(-\pi, \pi]$, we will called this unique value to be the **principal argument** of z and denote

it by $\text{Arg}(z)$. And the argument of z is defined by the set :
 $\arg(z) = \{ \text{Arg}(z) + 2k\pi : k \text{ is an integer} \}$.

Remark 2 : We can easily obtain the fact that $|zw| = |z||w|$ and $\arg(zw) = \arg(z) + \arg(w)$. But it is not true that $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$. (try the example $z = i$ and $w = -1$)

Theorem 2. (*Triangle inequality*) Suppose that z_1, \dots, z_n are complex numbers, then it holds

$$|z_1 + \dots + z_n| \leq |z_1| + \dots + |z_n|.$$

0.2 Some Geometric Objects Represented In Complex Theory

The parametric representation of a straight line passing through the point a and having the direction v is given by $z = a + tv$ where t is a real number.

The parametric representation of a straight line passing through the point a and b is given $z = (1-t)a + tb$ where t is a real number. If we only consider the line segment from a to b , the parametric representation of this line segment is given by $z = (1-t)a + tb$ with $t \in [0, 1]$.

If the straight line is a perpendicular bisector of the line segment joining a and b , then the set representation of this straight line is given by $\{z \in \mathbb{C} : |z - a| = |z - b|\}$.

The set representation of the straight line passing through the point a and b is given by $\left\{ z \in \mathbb{C} : \text{Im} \left(\frac{z - a}{a - b} \right) = 0 \right\}$.

The parametric representation of a circle centred at the point a and having radius $r > 0$ is given by $z = a + re^{i\theta}$ where $\theta \in [0, 2\pi)$.

The set representation of a circle centred at the point a and having radius $r > 0$ is given by $\{z \in \mathbb{C} : |z - a| = r\}$.

The set representation of a circle passing through the point a, b and c is given by $\left\{ z \in \mathbb{C} : \text{Im} \left(\frac{z - a}{z - b} \cdot \frac{c - b}{c - a} \right) = 0 \right\}$.

Proposition 3. (*The roots of unity*) If $z^n = a = re^{i\theta}$, then $z = r^{1/n} e^{i(\frac{\theta + 2k\pi}{n})}$ for $k = 0, 1, 2, \dots, n - 1$.

Definition 3. Let $\Omega \subset \mathbb{C}$, Ω is an open set if for any $z_0 \in \Omega$ there is a $r > 0$ such that $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\} \subset \Omega$.

Definition 4. Let $T \subset \mathbb{C}$, T is a closed set if its complement set $\mathbb{C} \setminus T$ is open.

Exercise : 1. Prove that $|zw| = |z||w|$ and $\arg(zw) = \arg(z) + \arg(w)$. Also give a example to disprove $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$

2. Show that $\text{Re}(z) \leq |z|$ and $\text{Im}(z) \leq |z|$.

3. Express $(1 + i)^5$ in standard form.

4. Prove the inverse triangle inequality: $||z| - |w|| \leq |z - w|$.